

# Rotating De Sitter Space

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## Abstract

An exact solution of the vacuum Einstein equations with a cosmological constant is exhibited which can perhaps be used to describe the interior of compact rotating objects. The physical part of this solution has the topology of a torus, which may shed light on the origin of highly collimated jets from compact objects.

*Key words:* de Sitter, rotating space-times, black holes

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## 1 Introduction

In this paper we exhibit an exact solution to the Einstein field equations that may help resolve two outstanding puzzles in theoretical astrophysics. The first puzzle is to describe the nature of space-time inside a rotating object that is sufficiently compact that it lies entirely inside a surface where classical general relativity predicts that an event horizon would form. The conventional view is that such an object is a “black hole”. However, both non-rotating and rotating black holes have features such as singularities and “reversal of space and time” that may be unphysical. In addition, the Kerr solution for a rotating black hole [1] shares in common with other rotating solutions of Einstein’s equations the pathological feature that there are closed time-like curves. It has been pointed out [2,3] that in the case of non-rotating compact objects the objectionable features of the non-rotating black hole interior space-time would be removed if the interior Schwarzschild space-time were replaced with de Sitter’s “interior” cosmological solution [4]. This space is non-singular and removes the “reversal

of space and time” that plagues the interior Schwarzschild solution. Furthermore the de Sitter interior solution can be made to exactly match the exterior Schwarzschild solution at the event horizon if the vacuum energy is chosen so that the total mass-energy of the interior de Sitter solution matches the black hole mass. According to this new picture of non-rotating compact objects space-time would not be analytically smooth at the event horizon, so classical general relativity would fail there. However, it has long been recognized that quantum effects become important near an event horizon, and therefore it is quite plausible that classical general relativity fails in the vicinity of an event horizon. References [2] and [3] offer two different scenarios as to what actually happens at the event horizon. However, for the purposes of this paper it not necessary to understand in detail what happens at the event horizon; instead we will focus on question as to whether there is a candidate space-time that could serve as a non-singular model for the bulk interior space-time inside rotating compact objects.

In accordance with the expectation that the interior space-time of a collapsed object should be obtained by continuous “squeezing” of a condensate vacuum state [5], we expect that this space-time should have a large vacuum energy; i.e. this interior space-time should locally resemble classical flat space-time with a cosmological constant; i.e. it should locally look like a region of de Sitter space-time. One nagging question concerning the proposals of references [2] and [3], though, is what should replace de Sitters interior solution in the case of a rotating compact object. To our knowledge a rotating version of de Sitter space-time has never been explicitly discussed in the literature. In the following we address this deficiency by exhibiting a mathematically exact solution of Einstein’s field equations that is in fact a rotating generalization of de Sitter’s interior solution. Not only does this solution provide a plausible picture for the nature of space-time in the interior of rotating compact objects, but as a bonus this solution provides a new insight into the nature of the highly collimated jets that have been observed to be emanating from compact astrophysical objects.

## 2 The metric

Our proposed interior metric is (we use units such that  $8\pi G/c^2 = 1$ )

$$ds^2 = \left[1 - \frac{\Lambda}{3}(r^2 - a^2 \cos^2 \theta)\right] dt^2 + 2a \left[1 - \frac{\Lambda}{3}r^2 \cos^2 \theta\right] dt d\varphi - \frac{r^2 + a^2 \cos^2 \theta}{r^2 - \frac{\Lambda}{3}r^4 + a^2} dr^2 - \frac{r^2 + a^2 \cos^2 \theta}{1 - \frac{\Lambda}{3}a^2 \cos^2 \theta \cot^2 \theta} d\theta^2 - (r^2 \sin^2 \theta - a^2 \cos^2 \theta) d\varphi^2 \quad (1)$$

where  $a$  is the angular momentum per unit mass. This metric is a limiting case of a class of metrics discovered by Carter [6] and independently by Plebanski [7]. In the limit  $a \rightarrow 0$  the metric (1) reduces to de Sitter's 1917 metric. It is clear by inspection that in contrast with the interior Kerr metric there are no space-time singularities near to  $r = 0$  for any value of  $\theta$ . The apparent singularities in the  $g_{rr}$  and  $g_{\theta\theta}$  components of the metric tensor can be removed by a change of variables [7], and represent event horizons where  $g_{00}g_{\varphi\varphi} - g_{0\varphi}^2 = 0$ . The singularity in  $g_{rr}$  is associated with a spherical event horizon located at

$$r_H^2 = \frac{3}{2\Lambda} + \left[\frac{9}{4\Lambda^2} + \frac{3a^2}{\Lambda}\right]^{1/2} \quad (2)$$

In the limit  $a \rightarrow 0$   $r_H$  becomes the de Sitter horizon  $\sqrt{3/\Lambda}$ . In addition to the spherical event horizon (2) there is a conical event horizon located at

$$\tan^2 \theta_H = -\frac{1}{2} + \sqrt{\frac{\Lambda}{3}a^2 + \frac{1}{4}} \quad (3)$$

In the case of slow rotation  $\Lambda a \ll 1$  this conical event horizon is located very near to the axis of rotation.

The nemesis of rotating space-times, closed time-like curves, will appear if  $g_{\varphi\varphi}$  is positive for some values of  $r$ . In our case this means that

$$r^2 \sin^2 \theta - a^2 \cos^2 \theta < 0 \quad (4)$$

This is will be satisfied if  $\varrho < a$ , where  $\varrho$  is the horizontal distance from the axis of rotation. Thus for slow rotation closed time-like curves will appear very close to the rotation axis. This is very reminiscent of the situation with space-time spinning strings [8]. Actually for all values of the rotation parameter  $a$  the conical horizon (3) lies inside the region where closed time-like curves

appear, and the critical angle where the inequality in (4) becomes an equality is precisely the horizon angle  $\theta_H$  where  $r = r_H$ .

The event horizon for the Kerr solution will match the event horizon (2) if the mass parameter for the Kerr solution is

$$m = \frac{\Lambda}{6} r_H^3 \quad (5)$$

Curiously this is the same condition that was used in ref. [2] to match de Sitter's interior solution to the exterior Schwarzschild solution in the case of a non-rotating compact object. If we impose the condition (5) the event horizon for our "interior" solution will occur at precisely the same radius as the horizon for the Kerr solution. In this case it might be reasonable to suppose that the "exterior" space-time outside the horizon (2) is just the usual exterior Kerr solution. Near to the spherical event horizon (2) the angular part of our rotating metric (1) has the form

$$ds^2 = -a^2 \frac{\sin^2 \theta - \frac{\Lambda}{3} a^2 \cos^4 \theta}{r_H^2 + a^2 \cos^2 \theta} \left( dt - \frac{r_H^2}{a} d\varphi \right)^2 - \frac{r_H^2 + a^2 \cos^2 \theta}{1 - \frac{\Lambda}{3} a^2 \cos^2 \theta \cot^2 \theta} d\theta^2. \quad (6)$$

For comparison the angular part of the Kerr metric when expressed in Boyer-Lindquist coordinates [9] and evaluated on the event horizon is

$$ds^2 = -a^2 \frac{\sin^2 \theta}{r_H^2 + a^2 \cos^2 \theta} \left( dt - \frac{r_H^2}{a} d\varphi \right)^2 - (r_H^2 + a^2 \cos^2 \theta) d\theta^2. \quad (7)$$

It can be seen that except near to  $\theta = 0$  the angular part of our metric near to the spherical event horizon is not too different to the angular part of the Kerr metric at the event horizon for all values of  $a$  such that  $\Lambda a^2 < 1$ . Significant difference do appear near to  $\theta = 0$ , which as we discuss below is due to the appearance of new physics near to the axis of rotation.

Inside the spherical event horizon (2) the behavior of our metric is completely different from that of the interior Kerr metric. For example, there are no space-time singularities. In the case of the Kerr solution  $g_{00} < 0$  everywhere inside the "ergosphere" whose outer boundary lies outside the event horizon at  $r^2 + a^2 \cos^2 \theta = 2mr$ . Although our  $g_{00}$  is negative at the event horizon (and close to the Kerr  $g_{00}$ ), it is actually positive for all values of  $r$  inside  $r^2 = (3/\Lambda)^{1/2} + a^2 \cos^2 \theta$ , which for small  $\Lambda a^2$  would be almost everywhere in the interior. In addition in contrast with the Kerr solution the radial metric coefficient  $g_{rr}$  is negative for all values of  $r$  inside the spherical event horizon. Thus the problematic reversal of the roles of time and radial distance in the interior Kerr solution is alleviated. The Kerr solution has the property that

inside the ergosphere particles cannot be at rest but must rotate about the axis. At the event horizon the frame in which particles could be at rest rotates with the “frame dragging” angular velocity

$$\left. \frac{d\varphi}{dt} \right|_{r=r_H} = \frac{a}{r_H^2 + a^2}. \quad (8)$$

For the metric (1)  $g_{00} < 0$  at the spherical event horizon and the frame rotation velocity is  $a/r_H^2$ , so particles in our space-time will also rotate as they approach the event horizon from the inside. Indeed our interior metric contains a reflection of the usual exterior Kerr ergosphere, with an inside boundary at  $r^2 = (3/\Lambda)^{1/2} + a^2 \cos^2 \theta$ . Thus in our picture of rotating compact objects the metric just inside the event horizon is a reflection of the metric just outside, at least away from  $\theta = 0$ . Reflection symmetry between the inner and outer metrics at an event horizon is just the matching condition for metrics suggested in ref. [2], and is a consequence of replacing the smooth geometry at an event horizon that is predicted by classical general relativity with a quantum critical layer.

### 3 Comparison with Demianski and Plebanski metrics

One might guess that the metric (1) could also be derived from Demianski’s well known generalization of the Kerr solution to include a cosmological constant [10]. Indeed taking the  $m = 0$  limit of Demianski’s metric yields

$$\begin{aligned} ds^2 = & \left[ 1 - \lambda(r^2 + a^2 \sin^2 \theta) \right] dt^2 + 2a \sin^2 \theta \cdot \lambda(r^2 + a^2) dt d\varphi \\ & - \frac{r^2 + a^2 \cos^2 \theta}{(r^2 + a^2)(1 - \lambda r^2)} dr^2 - \frac{r^2 + a^2 \cos^2 \theta}{1 + \lambda a^2 \cos^2 \theta} d\theta^2 \\ & - (a^2 + r^2) \sin^2 \theta \left[ 1 + \lambda a^2 \right] d\varphi^2, \end{aligned} \quad (9)$$

where  $\lambda = \frac{\Lambda}{3}$ . As was the case for metric (1) the variables  $\theta$  and  $\varphi$  represent the polar angles on a sphere. At first sight (1) and (9) appear to be different. However, both metrics (1) and (9) can be obtained as special cases of the Plebanski metric [7], which has the general form

$$\begin{aligned} ds^2 = & \frac{\mathcal{Q}}{p^2 + q^2} (dt - p^2 d\sigma)^2 - \frac{\mathcal{P}}{p^2 + q^2} (dt + q^2 d\sigma)^2 \\ & - \frac{p^2 + q^2}{\mathcal{P}} dp^2 - \frac{p^2 + q^2}{\mathcal{Q}} dq^2 \end{aligned} \quad (10)$$

When the mass, NUT charge, and electric and magnetic charges are all zero then the functions  $\mathcal{P}(p)$  and  $\mathcal{Q}(q)$  have the simple forms  $\mathcal{P} = b - \epsilon p^2 - \lambda p^4$  and  $\mathcal{Q} = b + \epsilon q^2 - \lambda q^4$ . In this case the Weyl tensor vanishes and the Plebanski metrics are conformally flat. For both metrics (1) and (9)  $p = a \cos \theta$ ,  $q = r$ ,  $\sigma = \varphi/a$ , and  $b = a^2$ . However, for metric (1)  $\epsilon = 1$  and  $\tau = t$ , while for the metric (9)  $\epsilon = 1 - \lambda a^3$  and  $\tau = t + a^2 \sigma$ .

Evidently the essential difference between the two geometries lies in the value of  $\epsilon$ . However it is known that Plebanski metrics with different values of  $\epsilon$  are related by a certain scaling transformation. This scaling transformation has the form

$$p' = p/\alpha, \quad q' = q/\alpha, \quad \sigma' = \alpha^3 \sigma, \quad \tau' = \alpha \tau, \quad b' = b/\alpha^4, \quad \epsilon' = \epsilon/\alpha^2, \quad (11)$$

where  $\alpha$  is the scaling parameter. The value of  $\Lambda$  is unchanged. Because  $b = a^2$  for both metrics we may replace  $a$  by  $\sqrt{b}$  so that both metrics depend only on the parameters  $b$  that appear in the original Plebanski metric (10). That the metrics (1) and (9) are locally isometric can now be seen as follows: we start with the  $m = 0$  Demianski metric (9) and rescale it using the scaling transformation (11) and  $\alpha = 1 - \lambda a^2$ . This leads to the metric (1) with  $\epsilon = 1$  and  $a^2 = b/(1 - \Lambda b/3)$ . Therefore the  $m = 0$  Demianski's geometry is locally isometric to the geometry of (1). The question to whether the  $m = 0$  Demianski geometry is isomorphic to our rotating solution is more complicated because the value of  $b$  affects the ranges of  $p$  and  $\varphi$ . In particular since  $\varphi = 0$  is identified with  $\varphi = 2\pi$ , then in the scaled metric  $\sigma = 0$  is identified with  $\sigma = 2\pi/\sqrt{b}$ . In addition the range of  $p$  is restricted because  $p \in [\sqrt{b}, +\sqrt{b}]$ . Thus whereas the metric (1) is defined for all values of  $\theta$  and  $\varphi$  the  $m = 0$  Demianski geometry corresponds to only a part of the sphere (cf. Fig. 1). Amusingly the latitudes covered by the  $m = 0$  Demianski are just those outside the conical horizon eq. (3). In accordance with our a priori expectations regarding the nature of the vacuum state inside compact objects, both metrics are locally isometric to de Sitter space-time.

## 4 Behavior near to the axis of rotation

As noted above the metric (1) is plagued by time-like closed curves near to the axis of rotation. Closed time-like curves are extremely pathological from the point of view of quantum mechanics. The pathological nature of the space-time corresponding to the metric (1) near to the axis of rotation can also be seen from the signature of the metric. Physical space-times should have the signature  $+- --$ . In the case of the Plebanski metrics this is only possible if  $\mathcal{P} > 0$ ,  $\mathcal{Q} > 0$  or  $\mathcal{P} > 0$ ,  $\mathcal{Q} < 0$ . If  $\mathcal{P} < 0$ , as is the case inside the conical

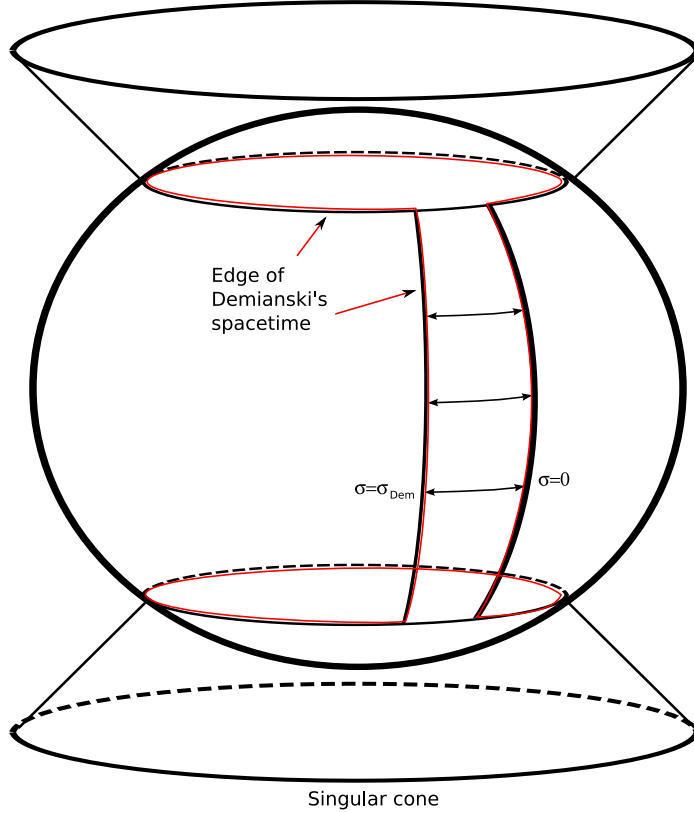


Fig. 1. The region of the spacetime (1) isometric (covered) by the Demianski's spacetime (9).

horizon, the signature is  $+++ -$ , so the space-time is not physical. Both of these considerations suggest that space-time undergoes some sort of phase transition near to the axis of rotation.

Recently it has been suggested [11] that the way to resolve the difficulties with classical rotating space-times that have closed time-like curves is to suppose that the rotation is actually carried by space-time “spinning strings” [8], in a manner analogous to the way rotation of superfluid helium inside a rotating container is carried by quantized vortices. The spinning strings resolve the question of the consistency of rotating space-times with quantum mechanics because the vorticity of space-time would be concentrated into the cores of the spinning strings where the condensate density would be very low and the Einstein equations are modified by the appearance of torsion. As shown in ref. [11] averaging over the vorticity of many perfectly aligned spinning strings leads to a Godel-like space-time. In a similar way it is reasonable to guess that the correct physical picture for the space-time inside the region where closed time-like curves appear in our solution for a rotating compact object is a Godel-like space-time. Indeed the Som-Raychaudhuri metric exhibited in ref. [11] may be a good approximation for the metric in this region. This metric would be applicable inside the critical radius where the local speed

of frame rotation is equal to the speed of light. The equation of motion for particles in a Godel-like space-time is well known [12]. In general this flow of particles will be collimated since the particles are confined to lie inside the cylinder where the velocity of frame rotation is less than the speed of light. In our situation the radius of this cylinder will equal the angular momentum per unit mass parameter  $a$  used in eq. (1); i.e. where the closed time-like curves first appear in our solution as the axis of rotation is approached. On the other hand particles in a Godel-like space-time are free to move parallel to the axis, so that for slow rotation of our compact object the flow of particles along the axis of rotation will be highly collimated.

## 5 Summary

In summary, the metric (1) provides an interior solution for rotating compact objects that avoids many of the unphysical features of the interior Kerr solution. It does not avoid the appearance of closed time-like curves, but the results of ref. [11] suggest that near to the axis of rotation a Godel-like phase of space-time appears where the vacuum energy is much smaller than in the bulk condensate of the rotating object and solid body-like rotation of the space-time appears.

Finally we should note that our metric (1) may also serve as a model for the large scale structure of our universe, where there are hints from observations of the large scale anisotropy of the cosmic microwave background that the universe might be rotating [13].

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## References

- [1] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).
- [2] G. Chapline, E. Hohfield, R. B. Laughlin, and D. Santiago. Phil. Mag. B **81**, pp. 235-254 (2001).
- [3] P. O. Mazur and E. Mottola, Proc. Nat. Acad. Sci. **111**, pp. 9545-9550 (2004).
- [4] W. de Sitter, Proc. Kon. Nat. Acad. Wet. **19**, pp. 1217-1255 (1917).
- [5] G. Chapline. in *Foundations of Quantum Mechanics* pp. 255-260, (World Scientific, 1992).
- [6] B. Carter, Phys. Lett. **26A**, pp. 399-400 (1968).



- [7] J. F. Plebanski, Ann. Phys. **90**, pp. 196-255 (1975).
- [8] P. O. Mazur, Phys. Rev. Lett. **57**, pp. 929-932 (1986).
- [9] R. H. Boyer and R. W. Lindquist, J. Math. Phys. **8**, 264 (1967).
- [10] M. Demianski, Acta Astronomica **23**, pp. 197-232 (1973).
- [11] G. Chapline and P. O. Mazur (gr-qc/0407033).
- [12] W. Kundt, Z. Phys. **C145**, 611
- [13] G. Chapline, astro-ph/0608389.